Tarski

$f : Pow(S) \to Pow(S)$ is *monotone* if $\forall S, S'. S \subseteq S' \Rightarrow f(S) \subseteq f(S')$.

For monotonic f, $\mu X.f(X) = \bigcap\{s \subseteq S \,|\, f(s) \subseteq s\}$ and $\nu X.f(X) = \bigcap\{s \subseteq S \,|\, s \subseteq f(s)\}$.

$f : Pow(S) \to Pow(S)$ is $\cup$-*continuous* if for all increasing chains X_i we have $\bigcup_{n\in\omega} f(X_n) = f(\bigcup_{n\in\omega} X_n)$.

$f : Pow(S) \to Pow(S)$ is $\cap$-*continuous* if for all decreasing chains X_i we have $\bigcap_{n\in\omega} f(X_n) = f(\bigcap_{n\in\omega} X_n)$.

For $\cup$-*continuous* f, $\mu X.f(X) = \bigcup_{n\in\omega} f^n(\emptyset)$.

For $\cap$-*continuous* f, $\nu X.f(X) = \bigcap_{n\in\omega} f^n(S)$.

For finite sets, since all chains are stationary all monotonic functions are continuous.

CCS

$p ::= nil \,|\, (\tau \to p) \,|\, (\alpha!a \to p) \,|\, (\alpha?x \to p) \,|\, (b \to p) \,| p_0 + p_1 \,|\, p_0||p_1 \,|\, p\backslash L \,|\, p[f] \,|\, P(a_1, \ldots, a_k)$

$(\tau \to p) \xrightarrow{\tau} p$

$$\frac{a \to n}{(\alpha!a \to p) \xrightarrow{\alpha!n} p}$$

$$\frac{}{(\alpha?x \to p) \xrightarrow{\alpha?n} p[n/x]}$$

$$\frac{b \to true \quad p \xrightarrow{\lambda} p'}{(b \to p) \xrightarrow{\lambda} p'}$$

$$\frac{p_0 \to p_0'}{p_0 + p_1 \xrightarrow{\lambda} p_0'}$$

$$\frac{p_1 \xrightarrow{\lambda} p_1'}{p_0 + p_1 \xrightarrow{\lambda} p_1'}$$

$$\frac{p_0 \xrightarrow{\lambda} p_0'}{p_0||p_1 \xrightarrow{\lambda} p_0'||p_1}$$

$$\frac{p_1 \xrightarrow{\lambda} p_1'}{p_0||p_1 \xrightarrow{\lambda} p_0||p_1'}$$

$$\frac{p_0 \xrightarrow{\alpha?n} p_0' \quad p_1 \xrightarrow{\alpha!n} p_1'}{p_0||p_1 \xrightarrow{\tau} p_0'||p_1'}$$

$$\frac{p_0 \xrightarrow{\alpha!n} p_0' \quad p_1 \xrightarrow{\alpha?n} p_1'}{p_0||p_1 \xrightarrow{\tau} p_0'||p_1'}$$

$$\frac{p \xrightarrow{\lambda} p' \quad \lambda \equiv \alpha?n \vee \lambda \equiv \alpha!n \Rightarrow \alpha \notin L}{pL \xrightarrow{\lambda} p'L}$$

$$\frac{p \xrightarrow{\lambda} p'}{p[f] \xrightarrow{f(\lambda)} p'[f]}$$

$$\frac{p[a_1/x_1, \ldots, a_k/x_k] \xrightarrow{\lambda} p'}{P(a_1, \ldots, a_k) \xrightarrow{\lambda} p'}$$

Pure CCS

$p ::= nil \,|\, \lambda.p \,|\, \sum_{i\in I} p_i \,|\, (p_0||p_1) \,|\, p\backslash L \,|\, p[f] \,|\, P \,|\, rec(P = p)$

$l ::= \alpha?n \,|\, \alpha!n$

$\lambda ::= l \,|\, \bar{l} \,|\, \tau$

$\lambda.p \xrightarrow{\lambda} p$

$$\frac{p_j \xrightarrow{\lambda} q \quad j \in I}{\sum_{i\in I} p_i \xrightarrow{\lambda} q}$$

$$\frac{p_0 \xrightarrow{\lambda} p_0'}{p_0||p_1 \xrightarrow{\lambda} p_0'||p_1}$$

$$\frac{p_1 \xrightarrow{\lambda} p_1'}{p_0||p_1 \xrightarrow{\lambda} p_0||p_1'}$$

$$\frac{p_0 \xrightarrow{l} p_0' \quad p_1 \xrightarrow{\bar{l}} p_1'}{p_0||p_1 \xrightarrow{\tau} p_0'||p_1'}$$

$$\frac{p \xrightarrow{\lambda} p'}{p[f] \xrightarrow{f(\lambda)} p'[f]}$$

$$\frac{p \xrightarrow{\lambda} q \quad P = p}{P \xrightarrow{\lambda} q}$$

$$\frac{p[rec(P = p)/P] \xrightarrow{\lambda} q}{rec(P = p) \xrightarrow{\lambda} q}$$

Hennesey-Milner Logic

$[\lambda]\, A = \neg \langle\lambda\rangle \neg A$

Finitary: $A ::= T \,|\, F \,|\, A_0 \wedge A_1 \,|\, A_0 \vee A_1 \,|\, \neg A \,|\, \langle\lambda\rangle\, A$

Infinitary: $A ::= \bigwedge_{i\in I} A_i \,|\, \neg A \,|\, \langle a\rangle\, A$

$p \asymp q \iff \forall A.(p \vDash A) \Leftrightarrow (q \vDash A)$

Modal μ-Calculus

$A ::= T \,|\, F \,|\, A \wedge B \,|\, A \vee B \,|\, \neg A \,|\, \langle a\rangle\, A \,|\, \langle . \rangle\, A \,|\, \nu X.A$

$\mu X.A = \neg \nu X.\neg A[\neg X/X]$

$$
\begin{array}{rcl}
S & = & S \text{ if } S \subseteq P \\
T & = & P \\
F & = & \emptyset \\
A \wedge B & = & A \cap B \\
A \vee B & = & A \cup B \\
\neg A & = & P \backslash A \\
\langle a \rangle A & = & \{p \in P | \exists q. p \xrightarrow{a} q \wedge q \in A\} \\
\langle . \rangle A & = & \{p \in P | \exists a, q. p \xrightarrow{a} q \wedge q \in A\} \\
\nu X.A & = & \bigcup \{S \subseteq P | S \subseteq A[S/X]\}
\end{array}
$$

CTL

$s \vDash EX\,A$ iff there exists a path from s whose next state satisfies A.

$s \vDash EG\,A$ iff there exists a path from s along which A holds globally.

$s \vDash E[A_0 U A_1]$ iff there exists a path from s where A_0 holds until A_1 does.

Local Model Checking

For monotonic f, $S \subseteq \nu X.f(X) \iff S \subseteq f(\nu X.(S \cup f(X)))$: this is the Reduction lemma.

The assertion $\nu X\{p_1, \ldots, p_n\}A$ denotes $\nu X.(\{p_1, \ldots, p_n\} \vee A)$.

$$
\begin{array}{rcl}
p \vDash S & = & p \in S \\
p \vDash T & = & true \\
p \vDash F & = & false \\
p \vDash \neg B & = & \neg_T (p \vDash B) \\
p \vDash A_0 \wedge A_1 & = & \\
p \vDash A_0 \vee A_1 & = & \\
p \vDash \langle a \rangle B & = & \\
\text{where } \{q_1, \ldots, q_n\} & = & \{q | p \xrightarrow{a} q\} \\
p \vDash \langle . \rangle B & = & \\
\text{where } \{q_1, \ldots, q_n\} & = & \{q | \exists a. p \xrightarrow{a} q\} \\
p \vDash \nu X\{\vec{r}\}B & = & true \text{ if } p \in \{\vec{r}\} \\
p \vDash \nu X\{\vec{r}\}B & = & p \vDash B[\nu X\{p, \vec{r}\}B/X] \text{ if } p \notin \{\vec{r}\}
\end{array}
$$

Bisimulation

A *strong bisimulation* is a binary relation R such that if pRQ: $(\forall \lambda, p', p \xrightarrow{\lambda} p' \Rightarrow \exists q'.q \xrightarrow{\lambda} q' \wedge p'Rq') \wedge (\forall \lambda, q'.q \xrightarrow{\lambda} q' \Rightarrow \exists p'.p \xrightarrow{\lambda} p' \wedge p'Rq')$.

$\sim = \bigcup\{R | R \text{ is a strong bisimulation}\} = \asymp$

If $p \sim \sum_{i \in I} \alpha_i.p_i$ and $q \sim \sum_{j \in J} \beta_j.q_j$ then $(p||q) \sim \sum_{i \in I} \alpha_i(p_i||q) + \sum_{j \in J} \beta_j(p||q_j) + \sum\{\tau.(p_i||q_j) | a_i = \bar{\beta}_j\}$

Petri Nets

General Nets

Have places/conditions P, transitions/events T, precondition map $pre : T \to mP$, postcondition map $post : T \to m^{\infty}P$ and capacity function $Cap \in m^{\infty}P$. A state is a marking M such that $M \leq Cap$.

$M \xrightarrow{t} M' \iff {}^{\cdot}t \leq M \wedge M' = M -^{\cdot} t + t^{\cdot}$

Basic Nets

Have conditions B, events E, precondition map $pre : E \to Pow(B)$, postcondition map $post : E \to Pow(B)$. A state is a marking M such that $M \subseteq B$.

$M \xrightarrow{\epsilon} M' \iff ({}^{\cdot}e \subseteq M \wedge (M \backslash {}^{\cdot}e) \cap e^{\cdot} = \emptyset) \wedge (M' = (M \backslash {}^{\cdot}e) \cup e^{\cdot})$

Have *contact* when ${}^{\cdot}e \subseteq M \wedge (M \backslash {}^{\cdot}e) \cap e^{\cdot} \neq \emptyset$, *safe* when contact cannot be reached

Basic Nets With Persistent Conditions

Augmented with subset of persistent conditions P.

$M \xrightarrow{\epsilon} M' \iff ({}^{\cdot}e \subseteq M \wedge (M \backslash (P \cup^{\cdot} e)) \cap e^{\cdot} = \emptyset) \wedge (M' = (M \backslash {}^{\cdot}e) \cup e^{\cdot} \cup (M \cap P))$

HOPLA

$\mathbb{P}, \mathbb{Q} ::= .\mathbb{P} \,|\, \mathbb{P} \to \mathbb{Q} \,|\, \sum_{a \in A} a\mathbb{P}_a \,|\, P \,|\, \mu_j \vec{P}\vec{\mathbb{P}}$

$t, u ::= x \,|\, rec\,x\,t \,|\, \sum_{i \in I} t_i \,|\, .\,t \,|\, [u > .x \Rightarrow t] \,|\, \lambda x\,t \,|\, t\,u \,|\, a\,t \,|\, \pi_a(t)$

Types

$$\frac{\Gamma(x) = \mathbb{P}}{\Gamma \vdash x : \mathbb{P}}$$

$$\frac{\Gamma, x : \mathbb{P} \vdash t : \mathbb{P}}{\Gamma \vdash rec\ x\ t : \mathbb{P}}$$

$$\frac{\Gamma \vdash t_j : \mathbb{P} \quad j \in I}{\Gamma \vdash \sum_{i \in I} t_i : \mathbb{P}}$$

$$\frac{\Gamma \vdash t : \mathbb{P}}{\Gamma \vdash .t : .\mathbb{P}}$$

$$\frac{\Gamma \vdash u : .\mathbb{P} \quad \Gamma, x : \mathbb{P} \vdash t : \mathbb{Q}}{\Gamma \vdash [u > .x \Rightarrow t] : \mathbb{Q}}$$

$$\frac{\Gamma, x : \mathbb{P} \vdash t : \mathbb{Q}}{\Gamma \vdash \lambda x\ t : \mathbb{P} \to \mathbb{Q}}$$

$$\frac{\Gamma \vdash t : \mathbb{P} \to \mathbb{Q} \quad \Gamma \vdash u : \mathbb{P}}{\Gamma \vdash t\ u : \mathbb{Q}}$$

$$\frac{\Gamma \vdash t : \mathbb{P}_b \quad b \in A}{\Gamma \vdash b\ t : \sum_{a \in A} a\mathbb{P}_a}$$

$$\frac{\Gamma \vdash t : \sum_{a \in A} a\mathbb{P}_a \quad b \in A}{\Gamma \vdash \pi_b(t) : \mathbb{P}_b}$$

$$\frac{\Gamma \vdash t : \mathbb{P}_j[\mu \vec{P}\vec{\mathbb{P}}/\vec{P}]}{\Gamma \vdash t : \mu_j \vec{P}\vec{\mathbb{P}}}$$

$$\frac{\Gamma \vdash t : \mu_j \vec{P}\vec{\mathbb{P}}}{\Gamma \vdash t : \mathbb{P}_j[\mu \vec{P}\vec{\mathbb{P}}/\vec{P}}$$

Actions

$p ::= \centerdot \mid u \mapsto p \mid a\, p$

$.\mathbb{P} : \centerdot : \mathbb{P}$

$$\frac{u : \mathbb{P} \quad \mathbb{Q} : q : \mathbb{Q}'}{\mathbb{P} \to \mathbb{Q} : (u \mapsto q) : \mathbb{Q}'}$$

$$\frac{\mathbb{P}_a : p : \mathbb{P}'}{\sum_{a \in A} a\mathbb{P}_a : a\ p : \mathbb{P}'}$$

$$\frac{\mathbb{P}_j[\mu \vec{P}\vec{\mathbb{P}}/\vec{P}] : p : \mathbb{P}'}{\mu_j \vec{P}\vec{\mathbb{P}} : p : \mathbb{P}'}$$

Transitions

$$\frac{\mathbb{P} : t[rec\ x\ t/x] \xrightarrow{p} t'}{\mathbb{P} : rec\ x\ t \xrightarrow{p} t'}$$

$$\frac{\mathbb{P} : t_j \xrightarrow{p} t' \quad j \in I}{\mathbb{P} : \sum_{i \in I} t_i \xrightarrow{p} t'}$$

$$\frac{}{\centerdot\mathbb{P} : \centerdot t \xrightarrow{\centerdot} t}$$

$$\frac{\centerdot\mathbb{P} : u \xrightarrow{\centerdot} u' \quad \mathbb{Q} : t[u'/x] \xrightarrow{q} t'}{\mathbb{Q} : [u > \centerdot x \Rightarrow t] \xrightarrow{q} t'}$$

$$\frac{\mathbb{Q} : t[u/x] \xrightarrow{p} t'}{\mathbb{P} \to \mathbb{Q} : \lambda x\ t \xrightarrow{u \mapsto p} t'}$$

$$\frac{\mathbb{P} \to \mathbb{Q} : t \xrightarrow{u \mapsto p} t'}{\mathbb{Q} : t\ u \xrightarrow{p} t'}$$

$$\frac{\mathbb{P}_a : t \xrightarrow{p} t'}{\sum_{a \in A} a\mathbb{P}_a : a\ t \xrightarrow{a\ p} t'}$$

$$\frac{\sum_{a \in A} a\mathbb{P}_a : t \xrightarrow{a\ p} t'}{\mathbb{P}_a : \pi_a(t) \xrightarrow{p} t'}$$

$$\frac{\mathbb{P}_j[\mu \vec{P}\vec{\mathbb{P}}/\vec{P}] : t \xrightarrow{p} t'}{\mu_j \vec{P}\vec{\mathbb{P}} : t \xrightarrow{p} t'}$$

Term t where $x : \mathbb{P} \vdash t : \mathbb{Q}$ is linear when $\forall I . t[\sum_{i \in I} u_i/x] \sim \sum_{i \in I} t[u_i/x]$