Linear Space

Set V of vectors is a linear space over field F of scalars if:

1. $\forall u,v \in V. u+v \in V$
2. $(u+v)+w = u+(v+w)$
3. $u+v = v+u$
4. $\exists \vec{0}. \vec{0}+v = v$
5. $\exists(-v). v+(-v) = \vec{0}$
6. $\forall v \in V, a \in F. av \in V$
7. $a(u+v) = au+av$
8. $(a+b)v = ab+bv$
9. $\exists 1. 1v = v$

Linear Subspace $W \subset V.\ \forall u,v \in W, a,b \in F$ have that $au+bv \in W$

Linear Combinat. u is a linear combination of $v_1, v_2, .. v_n$ if $\exists a_1, a_2 ... a_n . u = \sum_{i=1}^{n} a_i v_i$

Linear Independ. Vectors are linearly independent if $\sum_{i=1}^{n} a_i v_i = \vec{0} \leftrightarrow \forall i. a_i = 0$

Basis A set of vectors is a basis for a space V if they are linearly independent and their span is equal to V

Inner Product

Inner products must satisfy these:

1. $\langle v,v \rangle \geq 0$ and $\langle v,v \rangle = 0 \leftrightarrow v = \vec{0}$
2. $\langle au+bv, w \rangle = a\langle u,w \rangle + b\langle v,w \rangle$
3. $\langle u,v \rangle = \overline{\langle v,u \rangle}$

Norms

Norms on V must satisfy these:

1. $\|v\| \geq 0$ and $\|v\| = 0 \leftrightarrow v = \vec{0}$
2. $\forall a \in C. \|av\| = |a|\|v\|$
3. $\|u+v\| \leq \|u\| + \|v\|$

Cauchy-Schwarz $|\langle u,v \rangle|^2 \leq \langle u,u \rangle \langle v,v \rangle$

Orthogonality $u, v \text{ orthogonal} \leftrightarrow \langle u,v \rangle = 0$

Orthogonal System A sequence of vectors is orthogonal if all vectors are mutually orthogonal and not the zero vector

Orthonormal System An orthogonal system where $\|u_i\| = 1$

If $u = \sum_{i=1}^{n} a_i e_i$, $a_i = \langle u, e_i \rangle$

Fourier Coefficients Given a vector u, the scalars $\langle u, e_i \rangle$ wrt. an orthonormal system are called the Generalized Fourier coefficients of u

$\langle u,v \rangle = \sum_{i=1}^{n} \langle u, e_i \rangle \overline{\langle v, e_i \rangle}$

$\left\| \sum_{i=1}^{n} a_i u_i \right\|^2 = \sum_{i=1}^{n} |a_i|^2 \|u_i\|^2$

Orthogonal Projection $\tilde{u} = \sum_{i=1}^{n} \langle u, e_i \rangle e_i$

$\forall u \in V, w \in (W = span\{\text{ortho}\})$:

$\langle u - \tilde{u}, w \rangle = 0$,

$\|u-w\|^2 = \|u-\tilde{u}\|^2 + \|\tilde{u}-w\|^2$

The orthogonal projection is the closest vector to v in W

Closed An infinite orthogonal system is closed if $\|u - \tilde{u}\| = 0$

Complete An infinite orthogonal system is complete if the zero vector is the only one orthogonal to all basis vectors (and closure implies completeness)

Fourier Functions are piecewise continuous if they are continuous except at a finite number of points, and at these points the left and right finite limits exist

This is a linear space. Define:

$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx$

In the space of piecewise continuous functions,

$\{\frac{1}{\sqrt{2}}, \sin x, \cos x, \sin 2x, \cos 2x ..\}$

is a closed orthonormal sys.

Can also deal with piecewise continuous functions that map to complex numbers:

$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx$

$\{1, e^{ix}, e^{-ix}, e^{i2x}, ...\}$

The coefficients of these Fourier spaces are related by

$c_n = \frac{a_n - ib_s}{2}$, $c_{-n} = \frac{a_n + ib_s}{2}$, $c_0 = \frac{a_0}{2}$

Dirichlet Conditions A subspace of E where additionally $\forall x \in [-\pi, \pi)$ both left and right derivatives exist

Dirichlet's Theorem Given this, the Fourier series of a function converges pointwise to $\frac{f(x-)+f(x+)}{2}$ where f(x-), f(x+) are the left and right limits at $x \in [-\pi, \pi]$

Fourier Transform

$F(\omega) = F_{[f]}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$

Say functions f:R□C are piecewise continuous if they are piecewise continuous on every finite interval and absolutely integrable if $\int_{-\infty}^{\infty}|f(x)|dx < \infty$. Let G(R) be fns with both conditions
Subsequent properties will assume the functions in G(R), a linear space over the scalars C
F is defined everywhere and a continuous function

Inverse Fourier Transform

$\frac{f(x-)+f(x+)}{2} = \lim_{M\to\infty}\int_{-M}^{M} F_{[f]}(\omega)e^{iwx}d\omega$

Transform Properties

$F_{[f(ax+b)]}(w) = \frac{1}{a}e^{i\omega b/a}F_{[f]}(\frac{\omega}{a})$

$F_{[e^{icx}f(x)]}(w) = F_{[f]}(\omega - c)$

$F_{[f(x)\cos cx]}(w) = \frac{F_{[f]}(\omega-c)+F_{[f]}(\omega+c)}{2}$

Convolve

$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$

$F_{[f*g]}(\omega) = 2\pi F_{[f]}(\omega)F_{[g]}(\omega)$

Time Limited

If $f(x) = 0$ for all $|x| \geq M$

Band Limited

If $F_{[f]}(\omega) = 0$ for all $|\omega| \geq L$

Shannon Sampling Theorem

If f is band limited by L then

$f(x) = \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{L})\frac{\sin(Lx-n\pi)}{Lx-n\pi}$

Discrete Fourier Transform

$F[k] = \langle f, e_k \rangle = \sum_{n=0}^{N-1} f[n]e^{\frac{-2\pi i n k}{N}}$

(N point DFT of f[n])

Cyclical Convolution

$(f*g)[n] = \sum_{m=0}^{N-1} f[m]g[n-m]$

DFT of convolution is F[k]G[k]

Wavelet Transforms

$f(x) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} d_{jk}\psi_{jk}(x)$

$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$

An example of a mother wavelet is the Haar wavelet, a 1-oscillation square wave [0,1]
The coefficients d_{jk} give us information about f near the point $k2^{-j}$ on the scale 2^{-j}

Definitions

$X \sim U(0,1)$: X distribut. U(0,1)

$I(A)$: indicator function of A: $I(A)(w) = 1 \leftrightarrow w \in A$

F_X: probability distribution

f_X: probability density

Moment Generating Function

$M_X(t) = E(e^{tX})$, so:

$M_X(t) = 1 + E(X)t + E(X^2)\frac{t^2}{2!} + \ldots$

Markov's Inequality

$P(|X| \geq a) \leq \frac{E(|X|)}{a}$

Chebychev's Inequality

$P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$

Lyapunov's Inequality

$E(|X|^r)^{1/r} \geq E(|X|^s)^{1/s}$ ($r \geq s > 0$

Chernoff's Bound

$P(X \geq a) \leq e^{-ta}M(t)$

Convergence

In distribution:

$X_N \xrightarrow{D} X \text{ if } F_{X_n}(x) \to F_X(x)$ for all points x where F_X continuous

In probability:

$X_N \xrightarrow{P} X \text{ if } P(|X_n - X| > \varepsilon) \to 0$ for all $\varepsilon > 0$

Almost surely:

$X_N \xrightarrow{a.s} X \text{ if } P(X_n \to X) = 1$

In r^{th} mean:

$X_N \xrightarrow{r} X \text{ if } E(|X_n - X|^r) \to 0$

a.s. □ P, P □ D
If r>s≥1 then r □ s
If r≥1 then r □ P

Weak Law Of Large Numbers

Given n IID RVs with finite mean μ and variance σ^2:

$\overline{X_n} = \frac{\sum_i S_i}{n} \xrightarrow{P} \mu$

Strong Law Of Large Numbers

Given n IID RVs with finite mean μ and finite fourth moment: $\overline{X_n} \xrightarrow{a.s.} \mu$

Central Limit Theorem

Given n IID RVs with finite mean μ and variance σ^2 and whose mgf converges in some interval –a<t<a:

$Z_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} Z \sim N(0,1)$

Confidence Intervals

Define z_α so that $P(Z > z_\alpha) = \alpha$.
Now $\overline{X}_n \pm \frac{z_{\alpha/2}S}{\sqrt{n}}$ is an approximate confidence interval for the unknown μ

$\left(S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 \right)$

Markov Chains

Given n discrete RVs taking values in some countable S, then the sequence is a Markov chain if: $P(X_n = x_n \mid X_0 = x_0, .. X_{n-1} = x_{n-1})$ = $P(X_n = x_n \mid X_{n-1} = x_{n-1})$

Time homogenous if: $P(X_n = j \mid X_{n-1} = i) = P(X_1 = j \mid X_0 = i)$

Transition matrix is: $p_{ij} = P(X_n = j \mid X_{n-1} = i)$

N-step trans. matrix: $p_{ij}^{(n)} = P(X_n = j \mid X_0 = i)$

Chapman-Kolmogorov $p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)}$

Accessibility If for $n \geq 0, p_{ij}^{(n)} > 0$ then j accessible from i (i □ j)
If i □ j and j □ i then i □ j (they communicate)
A communicating class that once entered cannot be left is called closed
A closed communicating class of a single state is absorbing
A state space of a single communicating class is called irreducible, else reducible

Define f, the probability that starting in state I we visit state j for the first time at n:
$f_{ij}^{(n)} = P(X_1 \neq j, .., X_{n-1} \neq j, X_n = k \mid X_0 = i)$

If f_{ii}<1 then state i is transient, else recurrent

Mean Recurrence $T_j = \min\{n \geq 1 : X_n = j\}$, ∞ if no such visit occurs
$\mu_i = E(T_i \mid X_0 = i) = \sum_n n f_{ii}^{(n)}$ if i recurrent, ∞ otherwise
Positive recurrent if $\mu_i < \infty$, null recurrent otherwise

Periodicity Let $d_i = \gcd\{n : p_{ii}^{(n)} > 0\}$: if d_i = 1 then i is aperodic, else periodic with period d_i

Stationary Distribution $\pi = (\pi_j ; j \in S)$ is a stationary distribution with transition matrix P if: $\pi_j \geq 0$, $\sum_{j \in S} \pi_j = 1$, $\pi = \pi P$

Erdos-Feller-Pollard For an irreducible, aperiodic MC, if the MC is transient $p_{ij}^{(n)} \to 0$. If the MC is recurrent $p_{ij}^{(n)} \to \pi_j$ and if null recurrent $\pi_j = 0$ or positive recurrent π is a unique stationary distribution and $\mu_i = \frac{1}{\pi_i}$

Time Reversibility If X_n is an irreducible, positive recurrent MC with transition matrix P and unique stationary distribution π, $Y_n = X_{-n}$ is also an MC with stationary dist. π.
The MC is reversible if the trans. matrices are identical, and iff $\pi_i p_{ij} = \pi_j p_{ji}$ (the local balance conditions)