Definitions

Meta-variables: range over real objects
Interpretation: maps meta-variables to objects
Consistent: a set of statements is consistent if some interpretation satisfies them all
Entailment: a set of statements entails A if every interpretation that satisfies the statements in the sets satisfies A: S ≤ A
Validity: A is valid if every interpretation satisfies A: ≤ A
Equivalence: A and B are equivalent if A entails B and vice-versa: A > B
Deducibility: A deducible from S if there is a finite proof of A starting from S: S ′ A
Soundness: If S ′ A then S ≤ A
Completeness: If S ≤ A, S ′ A
Deduction: If S({A} ′ B then we can say that S ′ A □ B

Proposit. Identities

Note that we can get dual versions by swapping t,f,and,or

$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$

$(A \vee B) = \neg(\neg A \wedge \neg B)$

NNF

1. Get rid of implication
2. Push negation in

CNF

3. Push disjunctions in
4. Simplify (delete disjunction with P and !P, delete disjunction that includes another, replace $(P \vee A) \wedge (\neg P \vee A)$ by A)

This yields a theorem prover for propositional logic: does it simplify to a tautology?

Sequent Calculus

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{(cut)}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \text{(!L)}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \text{(!R)}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \text{(–L)}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \text{(–R)}$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \text{(/L)}$$

$$\frac{\Gamma \Rightarrow \Delta, A, B \quad \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A \vee B} \text{(/R)}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \text{(□L)}$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \text{(□R)}$$

First Order Logic

Allows reason about *functions* and *relations* over a *domain*
Function symbol: stands for an actual n-place function
Constant: 0-place functions
Variable: ranges over domain
Terms: variables, fn application
Relation symbol: stands for an actual n-place relation
Atomic formula: a relation applied to n terms
Formula: built from atomic formulae with propositional op.
Quantifiers: "for all" and "there exists"
Interpretation (D, I) consists of a domain D and a function I mapping symbols to real elements, fns and relations
Valuation V: gives values to free variables in a FOL formula

Truth

$\leq_{I,V}$ P(t) □ I[P](I_V[t])
$\leq_{I,V}$ u = v □ I_V[u] = I_V[v]
Others by obvious recursion
Validity: $\leq_I$ A □ $\leq_{I,V}$ A for all V
Satisfiable: A valid for some I

Substitution

With A[t/x] no variable of t must be bound in A

Identities

Get dual versions by swapping quantifiers, and, or

$\neg(\forall x A) = \exists x \neg A$

$(\forall x A) \wedge (\forall x B) = (\forall x A \wedge B)$

Holding only if x not free in B:

$(\forall x A) \wedge B = \forall x (A \wedge B)$

$(\forall x A) \vee B = \forall x (A \vee B)$

$(\forall x A) \rightarrow B = \exists x (A \rightarrow B)$

Sequent Calculus

$$\frac{A[t/x], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta} \text{(for all L)}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \forall x A} \text{(for all R,x NF con)}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta} \text{(exists L,x NF con)}$$

$$\frac{\Gamma \Rightarrow \Delta, A[t/x]}{\Gamma \Rightarrow \Delta, \exists x A} \text{(exists R)}$$

Clauses A disjunction of literals
Empty clause (□) means *f*

Method Prove A by contradiction:
1. Translate !A into CNF - this gives a clause set
2. Transform the clause set somehow
3. Deduce the empty clause (a contradiction)

DPLL
1. Delete tautologies such as {P, !P, …}
2. For each unit clause {L} delete all clauses containing L and delete !L from all others
3. Delete all clauses containing pure literals (i.e. assume that literal)
4. Perform a case split

Resolution $\dfrac{\{B, A_1,..,A_m\} \quad \{\neg B, C_1,...,C_n\}}{\{A_1,..,A_m,C_1,...,C_n\}}$
Combine this w/ unification to get "binary resolution" (rename variables apart in the clauses)

PNF
1. Convert to NNF
2. Push negation inside any quantifiers
3. Move quantifiers to the front

Skolemization
1. Convert to PNF
2. For every bound variable y in $\forall x_1,.., x_k \exists y A$ choose a new k-place function symbol f and replace y by f applied to the appropriate variables
3. Repeat until no exists quantifiers remain

Herbrand Universe For a clause set S:
$H_0 = \text{the set of constants in S}$ (must be non-empty: invent a constant to use if it is empty)
$H_{i+1} = H_i \cup \{ f(t_1,..,t_n) \mid t_1,..,t_n \in H_i\}$ for n-place function symbols in S
$H = \bigcup_{i \geq 0} H_i$ (the universe)

Herbrand Semantics Every constant stands for itself, every function symbol stands for a "term forming operation"
$HB = \{P(t_1,..,t_n) \mid t_1,..,t_n \in H\}$ for n-place predicate symbols in S
HB contains all ground atoms
A Herbrand model (a subset of HB) specifies the things we want to be true: this lets us construct a satisfying model syntactically

Herbrand's Theorem S in unsatisfiable □ there is a *finite* unsatisfiable set of *ground instances* of clauses of S

Unification Find a MGU by unifying terms in the tuple from left to right, applying the unification to terms yet to be unified at each step
Occurs check: cannot unify f(x) with x since x occurs in both

Factoring $\dfrac{\{B_1,...,B_n,A_1,..,A_m\}}{\{B_1,A_1,...,A_n\}\sigma}$ if $B_1\sigma = .. = B_n\sigma$

Prolog Clauses have at most 1 +VE literal
"Definite clause" with 1 +VE,0 -VE
"Goal clause" with 0 +VE, >0 -VE
To execute, resolve a program clause with goal clause repeatedly
Choose leftmost literal of goal clause and topmost definite clause
Depth first search, with backtrack
Unifies **without** occurs check!

BDDs Canonical form of expression
Sharing of identical subtrees, no redundant tests in the tree
Detects tautologies/inconsistency
Exhibits model if Satisfiable

Building Do not expand connectives e.g. iff
1. Convert operands to BDDs
2. Combine BDDS respecting the ordering and sharing
3. Delete redundant tests

To convert Z^Z':
1. Trivial case if either is t or f
2. Let:
    a. Z=if(P, X, Y)
    b. Z'=if(P', X', Y')
3. If P=P' recursively convert if(P, X^X', Y^Y')
4. If P<P' recursively convert if(P, X^Z', Y^Z')
5. If P>P' recursively convert if(P', Z^X', Z^Y')

Hash table optimisation: can do pointer compare for expressions

Variable order crucial (can be exp)

Modal Logic

Consists of (W,R) a set of worlds and an accessibility relation
Possibly: □, Necessarily: □ true in all *accessible* worlds
Interpretation: maps propositional letters to *subsets* of W
Universally valid: A such that $|\leq_{W,R}$ A for all frames (W, R): A is a tautology. Typically we restrain R

Truth

w |≤ □A □ v |≤ A for some v such that R(w, v)
Others by obvious recursion

Identities

□A = !□!A
□A □ A (reflexive: T)
□A □ □□A (transitive: S4)

Sequent Calculus

$$\frac{A,\Gamma \Rightarrow \Delta}{\Xi A,\Gamma \Rightarrow \Delta}$$ (□L)

$$\frac{\Gamma^* \Rightarrow \Delta^*, A}{\Gamma \Rightarrow \Delta, \Xi A}$$ (□R)

$$\frac{A,\Gamma^* \Rightarrow \Delta^*}{\lozenge A,\Gamma \Rightarrow \Delta}$$ (□L)

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \lozenge A}$$ (□R)

Γ* erases non-□ assumptions
Δ* erases non-□ goals

Tableaux Calculus

Work in the sequent calculus with only expressions in NNF
This means we only need one side of the sequent rules: choose left

Sequent Rules

$$\frac{}{A,\neg A,\Gamma \Rightarrow}$$ (basic)

$$\frac{\neg A,\Gamma \Rightarrow \quad A,\Gamma \Rightarrow}{\Gamma \Rightarrow}$$ (cut)

$$\frac{A,B,\Gamma \Rightarrow}{A \wedge B,\Gamma \Rightarrow}$$ (–L)

$$\frac{A,\Gamma \Rightarrow \quad B,\Gamma \Rightarrow}{A \vee B,\Gamma \Rightarrow}$$ (/L)

$$\frac{A[t/x],\Gamma \Rightarrow}{\forall x A,\Gamma \Rightarrow}$$ (for all L)

$$\frac{A,\Gamma \Rightarrow}{\exists x A,\Gamma \Rightarrow}$$ (exists L, x NF in Γ)

$$\frac{A,\Gamma \Rightarrow}{\Xi A,\Gamma \Rightarrow}$$ (□L)

$$\frac{A,\Gamma^* \Rightarrow}{\lozenge A,\Gamma \Rightarrow}$$ (□L)

Unification

For all on the left now inserts a *new* free variable: let unification instantiate any free variable: updating any variable should effect the *entire* proof tree.
Furthermore, skolemize "exists":

Skolemize

Push quantifiers *in*, not out
Then skolemize as usual