Propositional Logic

Grammar $a,b,c..| T | F | A \wedge B | A \vee B | \neg A$

Model $\|A\|_M \subseteq U_M$, $\|T\|_M = U_M$, $\|F\|_M = 0$

Validity A valid in M $\Leftrightarrow \|A\|_M = U_M \Leftrightarrow \leq$ A

A entails B $\Leftrightarrow \|A\|_M \subseteq \|B\|_M \Leftrightarrow$ A ≤ B

A ≤ B □ ≤ (A □ B)

≤ A □ $\leq_{TA}$ A, ≤ A □ A is a tautology

Functions

Function $\forall x, y, y'.(x, y) \in f(x) \wedge (x, y') \in f(x) \Rightarrow y = y'$

Total □ $\forall x \in X \,.\, f(x)$ is defined

Injective $\forall x, x' \in X. f(x) = f(x') \Rightarrow x = x'$

Surjective $\forall y \in Y. \exists x \in X. y = f(x)$

Bijective Bijective □ has an inverse

Images $RA = \{y \in Y \mid \exists x \in A.(x, y) \in R\}$

$R^{-1}A = \{x \in X \mid \exists y \in A.(x, y) \in R\}$

Relations

Equivalence $\forall x \in X.xRx$, $\forall x, y \in X.xRy \Rightarrow yRx$

$\forall x, y, z \in X.xRy \wedge yRz \Rightarrow xRz$

Class = $\{x\}_R = \{y \in X \mid yRx\}$

Partial Order $\forall p \in P.p \leq p$

$\forall p, q \in P.p \leq q \wedge q \leq p \Rightarrow q = p$

$\forall p, q, r \in P.p \leq q \wedge q \leq r \Rightarrow p \leq r$

Total □ all pairs comparable

A preorder lacks anti-symmetry

Lub X ⊆ P: u such that $\forall x \in X.x \leq u$

$\forall p \in P.(\forall x \in X.x \leq p) \Rightarrow u \leq p)$

Glb X ⊆ P: l such that $\forall x \in X.x \geq l$

$\forall p \in P.(\forall x \in X.x \geq p) \Rightarrow l \geq p)$

If a partial order has all lub, glb it is called a complete lattice

Set Size

Definition $|A| = |B| \Leftrightarrow A \cong B \Leftrightarrow$ A,B in bij. cor.

Finite $\Leftrightarrow \exists f. f : \{m \in \aleph \mid m \leq n\} \rightarrow A$

Countable A finite

A infinite $\wedge \exists f. f : \aleph \rightarrow A$

$\exists f, B \subseteq \aleph. f_{bij} : B \rightarrow A$

$\exists f. f_{inj} : A \rightarrow \aleph$

$\exists f, B$ countable. $f_{inj} : A \rightarrow B$

$\aleph \times \aleph$ is countable

Union of countable sets countable

$\Re$ is uncountable

Set Construction

Paradox $S = \{x \mid x \notin x\}$. Is S ∈ S?

Comprehension $\{x \in X \mid P(x)\}$

Powerset $P(X) = \{Y \mid Y \subseteq X\}$

Indexed Set $\{x_i \mid i \in I\}$

Disjoint Union $(\{1\} \times A\} \cup (\{2\} \times B)$

Miscellaneous Union, intersection, product, set difference

Foundation ∈ is well founded

Characteristic $X_Y : X \rightarrow \{T, F\} = (T \Leftrightarrow x \in Y)$

Size Sets are never in bijective correspondence with their powerset: this means there is no largest set

Inductive Definition

Rules Premise (X), axiom (y): X/y

R-closure $\Leftrightarrow \forall (X / y).X \subseteq Q \Rightarrow y \in Q$

Bounded □ exists R-closed set

$I_R = \bigcap \{Q \mid Q \text{ is R-closed}\}$

Induction $\forall (X / y) \in R.(\forall x \in X.x \in I_R \wedge P(x)) \Rightarrow P(y)$

Transitive Closure $R^+ = \{(a,b) \mid (a.b) \in R \vee ((a,c) \in R^+ \wedge (c,b) \in R^+)\}$

$R^* = R^+ \cup id_R$

Derivation $y \in I_R \Leftrightarrow \exists$R - derivation of y

Induction can occur on derivations

Well Founded Induction

Definition Relation □ such that there are no infinite descending chains of □

$a \leq b \Leftrightarrow a \prec b \vee a = b$

Relation □ on A well founded □

To Sets $\forall Q \subseteq A.(m \in Q \wedge \forall b \prec m.b \notin Q)$

Induction $\forall a \in A.((\forall b \prec a.P(b)) \Rightarrow P(a))$

Building Transitive closure retains w.f.

Product of two w.f.

Lexicographic product of two w.f

$f : A \rightarrow B, \prec_B \Rightarrow (a \prec_A a' \Leftrightarrow f(a) \prec_B f(a'))$

yields w.f. □$_A$ where □$_B$ is w.f.

Recursion Given B, □ on B well founded and a F mapping B and an arbitrary # of C to C for all inputs, we have:

$f(b) = F(b, f(b_1), .. f(b_n), ..), b_1 \prec .. \prec b_n \prec ..$

Where f is a unique total function